Instabilities in Collisionless Astrophysical Plasmas

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Vlasov Equation

Vlasov-Landau-Maxwell system (Schekochihin and Kunz (2016))

$$\frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \nabla f_s + \frac{Z_s e}{m_s} \left(\mathbf{E} + \frac{\mathbf{v} \times \mathbf{B}}{c} \right) \cdot \frac{\partial f_s}{\partial \mathbf{v}} = \left(\frac{\partial f_s}{\partial t} \right)_c$$

$$\cancel{\nabla \cdot} \mathbf{E} = 4\pi \sum_s Z_s e n_s, \qquad n_s \equiv \int \mathrm{d}^3\mathbf{v} f_s$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\frac{\partial \mathbf{B}}{\partial t} = -c \nabla \times \mathbf{E}$$

$$\nabla \times \mathbf{B} = \frac{4\pi}{c} \sum_s Z_s e n_s \mathbf{u}_s + \cancel{\frac{1}{c} \frac{\partial \mathbf{E}}{\partial t}}, \qquad \mathbf{u}_s \equiv \frac{1}{n_s} \int \mathrm{d}^3\mathbf{v}\, \mathbf{v} f_s$$

Vlasov Equation

Choice of coordinates

$$(\mathbf{r}, \mathbf{v}) \mapsto (\mathbf{r}, v_{\parallel}, \mathbf{w}_{\perp}) \mapsto (\mathbf{r}, v_{\parallel}, w_{\perp}, \varphi) \mapsto (\mathbf{r}, v_{\parallel}, \mu, \varphi)$$

Where (Parra (2016))

$$\frac{\mathrm{d}w_{\perp}}{\mathrm{d}t} \simeq \langle \dot{w}_{\perp} \rangle_{\varphi} = \frac{w_{\perp}}{2B}\frac{\mathrm{d}B}{\mathrm{d}T}$$

$$\mu \equiv \frac{w_{\perp}^2}{2B} \implies \frac{\mathrm{d}\mu}{\mathrm{d}t} \simeq 0$$

μ is an **adiabatic invariant**.

Vlasov Equation

In coordinates $(\mathbf{r}, v_\parallel, \mu, \varphi)$

$$\frac{DF_s}{Dt} + \left(\frac{Z_s e}{m_s} E_\parallel - \frac{D\mathbf{u}_\perp}{Dt} \cdot \hat{\mathbf{b}} - \underbrace{\mu \nabla_\parallel B}_{\text{mirror force}} \right) \frac{\partial F_s}{\partial v_\parallel} = \left(\frac{\partial F_s}{\partial t} \right)_c$$

Conserved kinetic energy along magnetic field line, $\mu B(\ell)$ a pseudopotential:

$$\mathcal{E} = \frac{1}{2} m^2 v_\parallel + m\mu B(\ell)$$

Thermal-magnetic energy balance; typically

$$\beta \equiv \frac{P}{B^2/2\mu_0} \geq 1$$

Anisotropy; take

$$F_s(\mathbf{r}, v_\parallel, \mu) \propto n_s(\mathbf{r}) \exp\left(-\frac{m_s v_\parallel^2}{2T_{s\parallel}} - \frac{m_s \mu B}{T_{s\perp}} \right)$$

Astrophysical Plasmas

Examples

Solar wind:

- Wide distribution of β; over 5 years, average $\log \beta = -0.04$ (Mullan and Smith (2006));
- $\beta_\perp - \beta_\| > 0$ in subsonic flow region of magnetosheath (Crooker et al. (1976)).

Comet tails:

- New ions occupy ring of phase space in comet frame (Coates (2004)).

Intracluster medium:

- $B \sim 10^{-6} G$;
- $\beta \sim 10^2$ (Melville et al. (2016))

β and Anisotropy in Instabilities

Following Melville et al. (2016) define

$$\Delta \equiv \frac{P_\perp - P_\parallel}{P} = \frac{T_\perp - T_\parallel}{T} = \frac{\beta_\perp - \beta_\parallel}{\beta}$$

Instability conditions:

- Mirror: $\Delta > \beta^{-1}$;
- Firehose: $\Delta < -2\beta^{-1}$.

Hot Plasma Waves

Let $\mathbf{B} = B\hat{\mathbf{z}}$. Look for perturbations $\propto \exp(i\mathbf{k}\cdot\mathbf{r} - i\omega t)$. Choose coordinates $\mathbf{k} = k_\perp \hat{\mathbf{x}} + k_\parallel \hat{\mathbf{z}}$. Dispersion relation

$$\left[\frac{c^2 k^2}{\omega^2}(\hat{\mathbf{k}}\hat{\mathbf{k}} - \mathbb{1}) + \boldsymbol{\epsilon}\right] \cdot \mathbf{E} = 0, \qquad \boldsymbol{\epsilon} = \mathbb{1} + \frac{i\boldsymbol{\sigma}}{\omega\epsilon_0}$$

In hot plasma limit

$$\frac{\omega}{k} \geq v_{\mathrm{ts}}$$

Hot Plasma Waves

Then (Parra (2016))

$$\boldsymbol{\epsilon} = \mathbb{1} + 2\pi \sum_s \frac{1}{n} \frac{\omega_{ps}^2}{\omega^2} \int_{C_L} \mathrm{d}v_\parallel \int_0^\infty \mathrm{d}v_\perp \left\{ \left(v_\perp \frac{\partial F_s}{\partial v_\parallel} - v_\parallel \frac{\partial F_s}{\partial v_\perp} \right) v_\perp \hat{\mathbf{z}}\,\hat{\mathbf{z}} \right.$$

$$\left. + v_\perp^2 \left[\frac{\partial F_s}{\partial v_\perp} + \frac{k_\parallel}{\omega} \left(v_\perp \frac{\partial F_s}{\partial v_\parallel} - v_\parallel \frac{\partial F_s}{\partial v_\partial} \right) \right] \sum_{n=-\infty}^{\infty} \frac{\omega \mathbf{u}_n \, \mathbf{u}_n^*}{\omega - k_\parallel v_\parallel - n\Omega_s} \right\}$$

$$\mathbf{u}_n = \frac{v_\parallel}{v_\perp} J_n(\lambda_s)\hat{\mathbf{z}} + \frac{n J_n(\lambda_s)}{\lambda_s}\hat{\mathbf{x}} - i J_n'(\lambda_s)\hat{\mathbf{y}}$$

$$\lambda_s = \frac{k_\perp v_\perp}{\Omega_s}$$

Following Hasegawa (1969), take limit

$$\frac{\omega}{\Omega_s} \ll 1, \quad \frac{k_\perp v_{\perp ts}}{\Omega_s} = \lambda_s \ll 1, \quad \frac{k_\parallel v_{\parallel ts}}{\Omega_s} \ll 1$$

Two dispersion relations

- $k^2 - \frac{\omega^2}{c^2}\left(\epsilon_{yy} + \frac{\epsilon_{yz}^2}{\epsilon_{zz}}\right) = 0$ (compressional modes);
- $k_\parallel^2 - \frac{\omega^2}{c^2}\epsilon_{xx} = 0$ (shear Alfvén wave).

Dispersion relation

$$k^2 - \frac{\omega^2}{c^2}\left(\epsilon_{yy} + \frac{\epsilon_{yz}^2}{\epsilon_{zz}}\right) = 0$$

In limit $\lambda_s \ll 1$, dominant contributions from $J_0(\lambda_s)$, $J_{\pm 1}(\lambda_s)$ so

- $\epsilon_{xy} = \epsilon_{yx} \simeq 0$;
- $\epsilon_{yz} = \epsilon_{zy} \simeq 0$;

and if we have hot ions with cold e^-

- $\frac{\omega^2}{c^2 k^2}\frac{\epsilon_{yz}^2}{\epsilon_{zz}} \ll 1$.

Mirror Instability

Hot Plasma Dispersion Derivation

Simplified dispersion relation

$$\implies k^2 = \frac{\omega^2}{c^2}\epsilon_{yy}$$

$$\epsilon_{yy} = \sum_s \frac{\omega_{ps}^2}{\Omega_s^2} - \frac{c^2}{\omega^2}\sum_s \left\{ k_\parallel^2 \frac{\beta_{\perp s} - \beta_{\parallel s}}{2} \right.$$
$$\left. + k_\perp^2 \beta_{\perp s}\left[1 + \frac{1}{2}\frac{T_{\perp s}}{T_{\parallel s}} Z'\left(\frac{\omega}{\sqrt{2}k_\parallel v_{\parallel ts}}\right)\right]\right\}$$

In low-frequency limit with hot ions, cold electrons

$$\frac{\omega}{\sqrt{2}k_\parallel v_{\parallel ts}} \ll 1, \qquad \frac{\omega^2}{k^2 v_A^2} \ll 1, \qquad v_A^2 \equiv \frac{B^2}{\mu_0 n_i m_i}$$

$$\omega = -ik_\parallel v_{\parallel ti}\sqrt{\frac{2}{\pi}}\frac{T_{\parallel i}}{T_{\perp i}\beta_{\perp i}}\left[\frac{k_\parallel^2}{k_\perp^2}\left(1 + \sum_{s \in \text{ ions}} \frac{\beta_{\perp s} - \beta_{\parallel s}}{2}\right) + 1 + \sum_{s \in \text{ ions}} \beta_{\perp s}\left(1 - \frac{T_{\perp s}}{T_{\parallel s}}\right)\right]$$

Mirror Instability

Hot Plasma Dispersion Derivation

Look for a wave with $k_{\parallel}^2 \ll k_{\perp}^2$, i.e. perpendicular propagation. With perturbations $\propto \exp(i\mathbf{k}\cdot\mathbf{r} - i\omega t)$ get instability condition

$$1 + \sum_s \beta_{\perp s}\left(1 - \frac{T_{\perp s}}{T_{\parallel s}}\right) < 0$$

with **purely imaginary** frequency

$$\omega = -ik_{\parallel} v_{\parallel ti}\sqrt{\frac{2}{\pi}}\frac{T_{\parallel i}}{T_{\perp i}\beta_{\perp i}}\left[1 + \sum_{s\in \text{ ions}} \beta_{\perp s}\left(1 - \frac{T_{\perp s}}{T_{\parallel s}}\right)\right]$$

Mirror Instability

Energetic Derivation

Following Southwood and Kivelson (1993), consider a plasma with one ion species and write the kinetic energy per iona f

$$W = W_{\|} + W_{\perp} \implies \delta W_{\|} = \delta W - \mu \delta B, \qquad \delta W_{\perp} = \mu \delta B$$

Time derivative of W:

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \mu \frac{\partial B}{\partial t} + \cancel{m v_{\|} \frac{\partial v_{\|}}{\partial t} + m \nabla_{\|} B v_{\|}}$$

Look for growing perturbations $\propto \exp(i k_{\|} v_{\|} + \gamma t)$ so

$$\mathrm{d}W = \mu \gamma B \, \mathrm{d}t \implies \delta W = \frac{\gamma}{\gamma + i k_{\|} v_{\|}} \mu \delta B$$

Consider

$$F(v_\parallel, \mathbf{w}_\perp)$$

Liouville's theorem gives

$$\begin{aligned}\delta F &= -\delta W_\parallel \frac{\partial F}{\partial W_\parallel} - \delta W_\perp \frac{\partial F}{\partial W_\perp} \\ &= \delta W_\parallel \frac{1}{T_\parallel} F + \frac{\mu \delta B}{T_\perp} F \\ &= \left[\frac{\delta W}{T_\parallel} + \mu \delta B \left(\frac{1}{T_\perp} - \frac{1}{T_\parallel}\right)\right] F\end{aligned}$$

Mirror Instability

Energetic Derivation

$$\delta F = \left[\frac{\mu \delta B}{T_\perp}\left(1 - \frac{T_\perp}{T_\parallel}\right)\right] F + \frac{1}{T_\parallel}\left(\frac{\gamma \mu \delta B}{\gamma + i k_\parallel v_\parallel}\right) F$$

Take second moment to get

$$\begin{aligned} 0 &= \frac{B^2}{2\mu_0} + P_\perp\left(1 - \frac{T_\perp}{T_\parallel}\right) + \gamma \frac{T_\perp^2}{T_\parallel}\int \mathrm{d}v_\parallel \, \frac{\gamma^2}{\gamma^2 + k_\parallel^2 v_\parallel^2} F_\parallel(v_\parallel) \\ &= \frac{B^2}{2\mu_0} + P_\perp\left(1 - \frac{T_\perp}{T_\parallel}\right) + \frac{\gamma}{|k_\parallel|}\frac{T_\perp^2}{T_\parallel}\pi \int \mathrm{d}v_\parallel \, \delta(v_\parallel) F_\parallel(v_\parallel) \end{aligned}$$

Rearrange:

$$-\frac{\gamma}{k_{\parallel}} = \frac{B^2}{2\mu_0}\frac{1+\beta_{\perp}(1-T_{\perp}/T_{\parallel})}{\pi(T_{\perp}^2/T_{\parallel})F_{\parallel}(0)}$$

Key features:

- $\gamma > 0 \iff 1+\beta_{\perp}\left(1-T_{\perp}/T_{\parallel}\right) < 0$;
- Depends inversely on $F_{\parallel}(0)$;
- Requires $k_{\parallel} \neq 0$.

Mirror Instability

Physical Interpretation

Particles with $v_\parallel \neq 0$ stream along field lines in mirrors created by perturbations (force $-\mu \nabla_\parallel B$), conserve μ, have MHD pressure response in antiphase to field (Hasegawa (1969)):

$$\delta P_\perp \sim 2 \underbrace{m\bar{n}v_\perp^2}_{P_\perp} \left(1 - \frac{T_\perp}{T_\parallel}\right) \frac{\delta B}{B}$$

Particles with $v_{\|} = 0$ only see perturbations, accelerate in phase with field increase:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}mw_{\perp}^{2}\right) = \mu\frac{\partial B}{\partial t}$$

Pressure response in phase with field.

Mirror Instability

Physical Picture

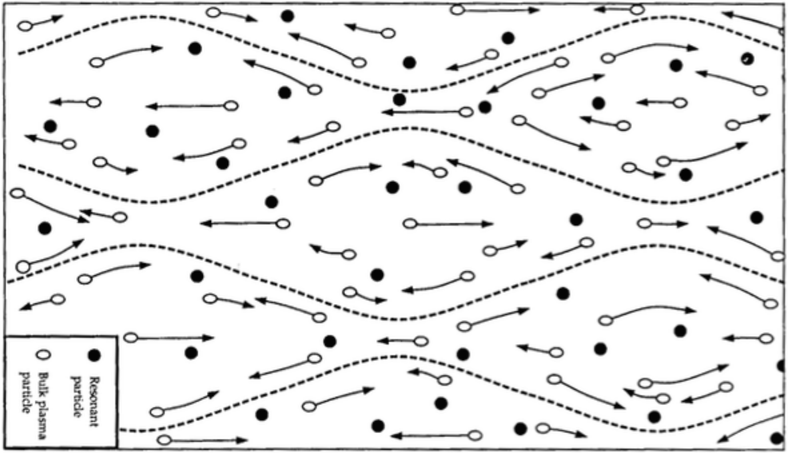

Figure 1: *Diagram of resonant and non resonant particles in the mirror instability (Southwood and Kivelson (1993))*

Firehose Instability

Hot Plasma Dispersion Derivation

Followng Hasegawa (1975), look for a shear wave with parallel propagation (i.e. magnetic field lines displaced perpendicular to themselves as before). Dispersion relation

$$k_{\parallel}^2 - \frac{\omega^2}{c^2}\epsilon_{xx} = 0$$

$$\begin{aligned}\epsilon_{xx} &= \sum_s \frac{\omega_{ps}^2}{\Omega_s^2}\left[\left\langle\left(1 - \frac{k_{\parallel} v_{\parallel s}}{\omega}\right)^2\right\rangle - \frac{k_{\parallel}^2\langle v_{\perp s}^2\rangle}{2\omega^2}\right] \\ &= \sum_s \frac{\omega_{ps}^2}{\Omega_s^2} - \frac{1}{2}\frac{c^2}{\omega^2}\sum_s k_{\parallel}^2(\beta_{\perp s} - \beta_{\parallel s})\end{aligned}$$

In low-frequency limit with hot ions, cold electrons

$$\frac{\omega^2}{k_\parallel^2 v_A^2} = 1 - \frac{1}{2}\sum_s (\beta_{\parallel s} - \beta_{\perp s})$$

Instability for opposite anisotropy to mirror instability

$$\sum_s T_{s\perp}\left(1 - \frac{T_{\parallel s}}{T_{\perp s}}\right) < -2$$

Equivalent to

$$\frac{B^2}{\mu_0} + \sum_s (P_{\perp s} - P_{\parallel s}) < 0$$

Kinetic MHD momentum conservation (Parra (2016))

$$n_s m_s \left(\frac{\partial \mathbf{u}_s}{\partial t} + \mathbf{u}_s \cdot \nabla \mathbf{u}_s \right) = -\nabla \cdot \mathbf{P}_s + Z_s e n_s (\mathbf{E} + \mathbf{u}_s \times \mathbf{B})$$

Integrate this equation over a flux tube around a magnetic field line to get

$$\mathbf{F} = \int_A \left(\frac{B^2}{2\mu_0} + \sum_s P_{\perp s} \right) (-\hat{\mathbf{n}}) \mathrm{d}^2 A + \int_{A_b} \left(\frac{B^2}{\mu_0} + \sum_s (P_{\perp s} - P_{\parallel s}) \right) \hat{\mathbf{n}} \mathrm{d}^2 A$$

Firehose Instability

Physical Picture

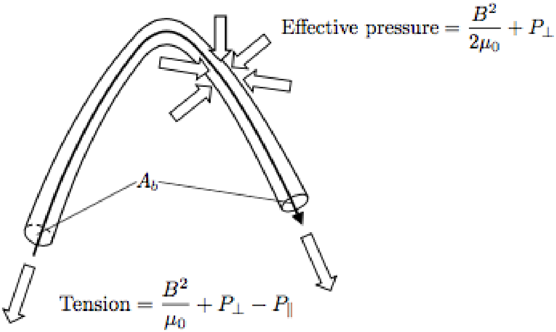

Figure 2: *Diagram of pressure forces on flux tube around magnetic field line (Parra (2016))*

Shearing, Relaxation (Melville et al. (2016))

Model turbulence in a plasma by shearing flow $\mathbf{u} = -Sx\hat{\mathbf{y}}$, alternating S.

Firehose:

- Perturbations scale as $\langle \left|\delta\mathbf{b}^2\right| \rangle \sim 2St$;
- Saturate at $\langle \left|\delta\mathbf{b}^2\right| \rangle \sim \sqrt{\beta S \Omega^{-1}}$.

Mirror:

- Perturbations scale as $\left|\delta B_{\parallel}/B_0\right| \sim (|S|\, t)^{2/3}$;
- Saturate at $\left|\delta B_{\parallel}/B_0\right| \sim 1$.

Shearing, Relaxation (Melville et al. (2016))

Relaxation relies on effective collisionality ν_{eff}:

- Firehose fluctuations decay as $\langle \left| \delta \mathbf{b}^2 \right| \rangle \propto e^{-\gamma t}, \gamma \sim \Omega/\beta$;
- Mirror perturbations decay as $\langle \delta B_{\parallel}^2 / B_0^2 \rangle (t) \propto (1 + (t - t_0)\gamma [\langle \delta B_{\parallel}^2 / B_0^2 \rangle (t_0)]^{1/4})^{-4}, \gamma \sim \Omega/\beta$.

Power-law decay is slower than exponential decay.
Can expect to drive alternating mirrors and firehoses by alternating the shear S.

Magnetogenesis (Melville et al. (2016))

Results

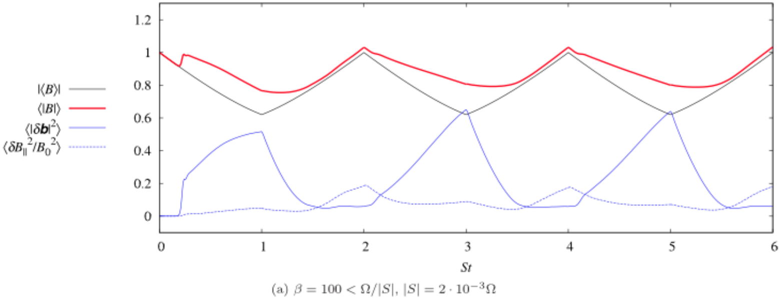

Figure 3: *Driving of firehose and mirror instabilities for low β ($\beta < \Omega/|S|$)*

Magnetogenesis (Melville et al. (2016))

Results

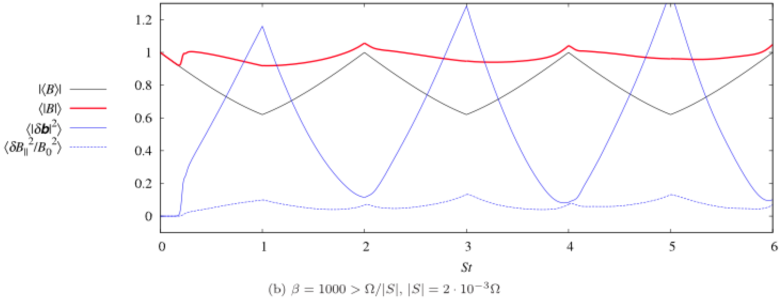

Figure 4: *Driving of firehoses and suppression of mirrors for high β ($\beta > \Omega/|S|$)*

Consequences

- Anisotropies are a generic feature of astrophysical plasmas (Crooker et al. (1976),Coates (2004));
- For high-β plasmas there is a fine range of anisotropies for which perturbations are stable;
- Turbulence in plasma can drive anisotropies towards and away from instabilities (Melville et al. (2016));
- Growth of small perturbations may explain cosmic magnetogenesis (Melville et al. (2016))

Outside this linear regime must consider FLR effects and collisionality.

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